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A CONJECTURE ON CONVOLUTION OPERATORS, AND A NON-DUNFORD–PETTIS OPERATOR ON L^1

BY

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ABSTRACT

There exists a non-Dunford–Pettis operator from L^1 into a Banach lattice E that does not contain a copy of c_0 or L^1. This problem is related to regularisation properties of convolution operators on L^1.

1. Introduction

H. P. Rosenthal proved that the "convolution by a biased coin" operator from L^1 into L^1 does not fix a copy of L^1 and fails the Dunford–Pettis property [4]. It is thus a natural question, raised by N. Ghoussoub (private communication), whether this can be improved by finding a non-Dunford–Pettis operator from L^1 into a Banach lattice E that does not contain c_0 or L^1. Observe that E must fail the Radon–Nikodym property, thus also improving an example of the author [5]. This problem is arguably not of the utmost importance. However, the natural approach raises more central problems, to be presently explained.

For $-1 \leqq a \leqq 1$, denote by μ_a the probability measure

$$\mu_a = \left(\left(\frac{1}{2} - \frac{a}{2}\right)\delta_{-1} + \left(\frac{1}{2} + \frac{a}{2}\right)\delta_1\right)^{\otimes \mathbf{N}}$$

on the group $\{-1, 1\}^{\mathbf{N}}$; thus $\mu_a * \mu_b = \mu_{ab}$. Denote $L^1 = L^1(\{-1, 1\}^{\mathbf{N}}, \mu_0)$, and

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denote by T_a the "convolution by a biased coin" operator $f \to \mu_a * f$ on L^1. If r_n denotes the nth coordinate function on $\{-1, 1\}^{\mathbf{N}}$, $T_a(r_n) = ar_n$, so that, for $a \neq 0$, T_a fails the Dunford–Pettis property, since it does not send the weakly convergent sequence (r_n) to a norm-convergence sequence. Since, as shown by Rosenthal, T_a does not fix a copy of L^1, the natural approach to Ghoussoub's problem is to try to factor $T_a : L^1 \to L^1$ through a Banach lattice E that does not contain L^1. It has been shown by N. Kalton [2] that when a Banach lattice E of measurable functions contains a copy of L^1, there exists a lattice isomorphism T of L^1 onto a sublattice of E. Then it is easily seen that if α is small enough that $|\{T(1) \geqq \alpha\}| > 0$, for each n there exists $0 \leqq k < 2^n$ such that

$$2^n |\{T(2^n 1_{[k2^{-n},(k+1)2^{-n}]}) \geqq 2^n\alpha\}| > |\{T1 \geqq \alpha\}|,$$

where $|A| = \mu_0(A)$.

Thus, a very natural way to ensure that E does not contain a copy of L^1 is to require that

$$\lim_{t\to\infty} t \sup\{|(f \geqq t\}|; f \in E, \|f\|_E \leqq 1\} = 0.$$

Since E has to contain the function $T_a f$ whenever $\|f\|_1 \leqq 1$, one is led to the following:

Problem 1. Is it true that, for $a \neq 1, -1$,

$$\text{(1)} \qquad \lim_{t\to\infty} t \sup\{|\{T_a f \geqq t\}|; f \in L^1, f \geqq 0, \|f\|_1 \leqq 1\} = 0?$$

In fact, by looking at examples, one is led to the following question:

Problem 2. Is it true that for $a \neq 1, -1$, there exists a constant $K = K(a)$ such that

$$\forall f \in L^1, \quad \|f\|_1 \leqq 1, \quad \forall t \geqq 2, \quad |\{T_a f \geqq t\}| \leqq K(a) t^{-1} (\log t)^{-1/2}?$$

It is a well-known fact that T_a is a "regularizing" operator. In particular, we have the hyper-contractivity property, for $q \geqq p > 1$,

$$\text{(2)} \qquad a \geqq \sqrt{\frac{q-1}{p-1}} \Rightarrow \|T_a f\|_q \leqq \|f\|_p \qquad \forall f \in L^p.$$

This does not tell us anything about the action of T on L^1; (1) would be a statement about the regularizing properties of T on L^1. Unfortunately, the condition of (1) is very much "non-convex" and the available machinery seems powerless to attack that question.

Being unable to answer either Problem 1 or Problem 2, we will turn towards the operator $T = \int_0^1 T_{e^{-u}} du$.

THEOREM 1. *We have*

$$\lim_{t\to\infty} t \sup\{ |\{Tf \geqq t\}| ; f\in L^1, f>0, \| f \|_1 \leqq 1\} = 0. \tag{3}$$

Actually, our proof shows that for t large enough and $f\in L^1$, $\| f \|_1 \leqq 1$, we have

$$|\{Tf \geqq t\}| \leqq K(\log\log t)/t \log t \qquad \text{where } K \text{ is a number.}$$

We have no reason to think this is sharp; the actual estimate is irrelevent for our purpose, and only (3) will be used for the proof of the following that answers Ghoussoub's question.

THEOREM 2. *The operator $T: L^1 \to L^1$ fails the Dunford–Pettis property, but factors through a Banach lattice E that does not contain c_0 or L^1.*

2. Proof of Theorem 1

Consider $f\in L^1$, $\| f \|_1 \leqq 1$, $f \geqq 0$, $t \geqq 1$. We set $h = \int_0^{1/N} T_{e^{-u}}(f) du$ where $N \geqq 1$ will be specified later. Since T_a is of norm one from L^1 to L^1, we have $\| h \|_1 \leqq 1/N$. Let $V = T_{e^{-1/N}}$. The formula $T_a \circ T_b = T_{ab}$ yields $T(f) = \Sigma_{i=0}^{N-1} V^i(h)$. For $0 \leqq i \leqq N-1$, we set $g^i = \min(V^i(h), t)$. We set $u^0 = g^0$, and for $1 \leqq i \leqq N-1$, we set $u^i = g^i - V(g^{i-1})$.

LEMMA 1. $\Sigma_{0\leqq i \leqq N-1} \| u^i \|_1 \leqq \| h \|_1 \leqq 1/N$.

PROOF. We have $g^i = V(g^{i-1}) + u^i$, so that $V^{N-i-1}(g^i) = V^{N-i}(g^{i-1}) + V^{N-i-1}(u^i)$. By summation of these equalities for $1 \leqq i \leqq N-1$, we get

$$g^{N-1} = V^{N-1}(g^0) + \sum_{i=1}^{N-1} V^{N-i-1}(u^i) = \sum_{i=0}^{N-1} V^{N-i-1}(u^i).$$

We observe that $\int V^{N-i-1}(u^i) d\mu_0 = \| u^i \|_1$, and that $\| g^{N-1} \|_1 \leqq \| V^{N-1}(h) \|_1 \leqq \| h \|_1 \leqq 1/N$. □

Set $U = \{T(f) \geqq t\}$. Since $T(f) = \Sigma_{i=0}^{N-1} V^i(h)$, we have

$$U \subset \left\{ \sum_{i=0}^{N-1} g^i \geqq t \right\} = \left\{ u^0 + \sum_{i=0}^{N-1} (u^i + V(g^{i-1})) \geqq t \right\}.$$

By integration over U, we get

$$t\,|U| \leqq \sum_{i=0}^{N-1} \|u^i\|_1 + \sum_{i=0}^{N-1} \int_U V(g^{i-1})\,d\mu_0. \tag{4}$$

Let now $p = 1 + e^{-1/N}$, $q = 1 + e^{1/N}$, so that $e^{1/N} = \sqrt{(q-1)/(p-1)}$. Hence, by (2), $\|V(f)\|_q \leqq \|f\|_p$ for all $f \in L^p$. From (4) and Lemma 1, we get

$$t\,|U| \leqq \frac{1}{N} + \sum_{i=1}^{N-1} |U|^{1-1/q} \|V(g^{i-1})\|_q$$

$$\leqq \frac{1}{N} + \sum_{i=1}^{N-1} |U|^{1-1/q} \|g^{i-1}\|_p.$$

Now

$$\|g^{i-1}\|_p \leqq \|g^{i-1}\|_1^{1/p} \|g^{i-1}\|_\infty^{1-1/p} \leqq N^{-1/p} t^{1-1/p}.$$

Thus we obtain

$$t\,|U| \leqq 1/N + (Nt)^{1-1/p} |U|^{1-1/q}.$$

Suppose now that N is the smallest integer with $N \geqq 2/t\,|U|$. Thus $1/N \leqq t\,|U|/2$, and thus

$$t\,|U| \leqq 2(Nt)^{1-1/p} |U|^{1-1/q}.$$

Since $t\,|U| \leqq 1$, we have $N \leqq 4/t\,|U|$, and thus

$$t\,|U| \leqq 2\left(\frac{4}{|U|}\right)^{1-1/p} |U|^{1-1/q} \leqq 8\,|U|^{1/p-1/q}.$$

Since $N \geqq 2$, we have

$$\frac{1}{p} - \frac{1}{q} = \frac{e^{1/N}-1}{e^{2/N}+1} \geqq \frac{1}{4N},$$

and so

$$t\,|U| \leqq 8\,|U|^{1/4N} \leqq 8\,|U|^{t|U|/16} \leqq 16\,|U|^{t|U|/16}.$$

We now set $y = t\,|U|/16 \leqq 1/16$. Hence $y \leqq (16y/t)^y \leqq (1/t)^y$, so that $\log y \leqq y \log(1/t)$, i.e., $y \log t \leqq \log(1/y)$. It follows easily that for t large enough, $y \leqq c(\log\log t)/\log t$, i.e., $|U| \leqq c \log\log t/t \log t$, where c is a universal constant. □

3. Proof of Theorem 2

We first give some notations. Let

$$\mathscr{C}_k = \{g \in L^1;\ \| g \|_1 \leqq 2^{-k},\ \exists f \subset L^1,\ \| f \|_1 \leqq 2^k, 0 \leqq g \leqq T(|f|)\}.$$

Let

$$\mathscr{C}' = \left\{ g = \sum_{k \geqq 1} \alpha_k g_k;\ \alpha_k \geqq 0,\ \sum_{k \geqq 1} \alpha_k \leqq 1, g_k \in \mathscr{C}_k \right\}.$$

We observe that if $g = \Sigma_{l \geqq 1} \alpha_l g_l$ where $g_l \in \mathscr{C}_{k(l)}$, we have $g \in (\Sigma_{l \geqq 1} \alpha_l)\mathscr{C}'$. Finally, let

$$\mathscr{C} = \{h \in L^1;\ \exists (h_n),\ h_n \in \mathscr{C}', 0 \leqq h \leqq \liminf h_n\}.$$

We observe that $\mathscr{C}_k$, hence $\mathscr{C}'$, hence $\mathscr{C}$ are convex. Moreover $\| g \|_1 \leqq 1$ for $g \in \mathscr{C}'$, hence for $g \in \mathscr{C}$. It is simple to see that we can define a Banach lattice E such that $\mathscr{C}$ is the positive unit ball of E, i.e.,

$$E = \{f \in L^1;\ \exists \lambda > 0,\ |f| \in \lambda \mathscr{C}\},$$

the norm of f being the infimum of such λ's. Since $\| g \|_1 \leqq 1$ for $g \in \mathscr{C}$, we have $\| f \|_1 \leqq \| f \|_E$. On the other hand, the definition of $\mathscr{C}$ shows that $T(L^1) \subset E$, and that $T: L^1 \to E$ is of norm $\leqq 1$. This shows that E factors through L^1.

The proof that E does not contain c_0 will use the following lemmas.

Lemma 2. *Let $h \in \mathscr{C}$. Then there exists a sequence (β_k), $\beta_k \geqq 0$, $\Sigma_{k \geqq 1} \beta_k \leqq 1$, and $g_{n,k} \in \mathscr{C}_k$ such that*

$$h \leqq \liminf_n \sum_{k \geqq 1} \beta_k g_{n,k}.$$

Proof. By definition of $\mathscr{C}$, we can find a sequence $h_n \in \mathscr{C}'$ such that $h \leqq \liminf_n h_n$. By definition of $\mathscr{C}'$, $h_n = \Sigma_{k \geqq 1} \alpha_{n,k} g_{n,k}$ where $\alpha_{n,k} \geqq 0$, $\Sigma_{k \geqq 1} \alpha_{n,k} \leqq 1$, $g_{n,k} \in \mathscr{C}_k$. There is no loss of generality to assume that $\beta_k = \lim_{n \to \infty} \alpha_{n,k}$ exists, and that moreover $|\beta_k - \alpha_{n,k}| \leqq 2^{-2k}$ for $n \geqq k$. We have $\Sigma_{k \geqq 1} \beta_k \leqq 1$. Set $h'_n = \Sigma_{k \leqq n} \beta_k g_{n,k}$. Thus we get

$$\| h_n - h'_n \|_1 \leqq \sum_{k \leqq n} |\beta_k - \alpha_{n,k}|\ \| g_{n,k} \|_1 + \sum_{k > n} \alpha_{n,k} \| g_{n,k} \|_1$$

$$\leqq n 2^{-n} + 2^{-n} = (n+1) 2^{-n}.$$

It follows that $\lim |h_n - h'_n| = 0$ a.e., and that $h \leqq \liminf_n h'_n$. □

LEMMA 3. *The norm of E is order continuous; that is, if for a sequence* $h_1 \geqq h_2 \geqq \cdots \geqq 0$ *such the* $\inf_n h_n = 0$, *we have* $\lim_{n\to\infty} \| h_n \|_E = 0$.

PROOF. By definition of E, it is clear that $\inf_n h_n = 0 \in E$ means that $\inf_n h_n = 0$ pointwise a.e. In particular $\| h_n \|_1 \to 0$. We can and do assume that $\| h_1 \|_E \leqq 1$. From Lemma 2, there exists a sequence $\beta_k \geqq 0$, $\Sigma_{k\geqq 1} \beta_k \leqq 1$, and $g_{n,k} \in \mathscr{C}_k$ such that

$$h_1 \leqq \liminf_n \sum_{k \geqq 1} \beta_k g_{n,k}.$$

To conclude the proof, it is sufficient to show that if $0 \leqq h \leqq h_1$ and $\| h \|_1 \leqq 2^{-2q}$, we have

$$\| h \|_E \leqq \alpha_q := 4q 2^{-q/2} + \sum_{k>q} \beta_k.$$

So we have to show that $h \in \alpha_q \mathscr{C}$. Set

$$u_n = \min\left(h, \sum_{k\geqq 1} \beta_k g_{n,k}\right).$$

Since $h \leqq \liminf_n u_n$, by definition of $\mathscr{C}$ it is enough to show that for each n we have $u_n \in \alpha_q \mathscr{C}'$. Since $u_n \leqq h$, we have $\| u_n \|_1 \leqq 2^{-2q}$. Since $u_n \leqq \Sigma_{k\geqq 1} \beta_k g_{n,k}$, we can write $u_n = \Sigma_{k\geqq 1} \beta_k g'_{n,k}$ where $g'_{n,k} \leqq g_{n,k}$. For $k \leqq q$ denote by s_k the largest integer such that $\| g'_{n,k} \|_1 \leqq 2^{-2s_k - k}$. Since $g_{n,k} \in \mathscr{C}_k$, we have $s_k \geqq 0$. Since $\| \beta_k g'_{n,k} \|_1 \leqq \| u_n \|_1 \leqq 2^{-2q}$, we have

$$2^{-2s_k-k} \leqq 4 \| g'_{n,k} \|_1 \leqq \beta_k^{-1} 2^{-2q+2},$$

so that

$$\beta_k 2^{-s_k} \leqq \beta_k^{1/2} 2^{-s_k} \leqq 2^{k/2-q+2}.$$

We have $\| 2^{s_k} g'_{n,k} \|_1 \leqq 2^{-s_k-k}$; the definition of $\mathscr{C}_k$ shows that $2^{s_k} g'_{n,k} \in \mathscr{C}_{k+s_k}$. We have

$$u = \sum_{k\leqq q} (\beta_k 2^{-s_k})(2^{s_k} g'_{n,k}) + \sum_{k>q} \beta_k g'_{n,k}.$$

This shows that $u \in \alpha \mathscr{C}'$, where

$$\alpha = \sum_{k\leqq q} (\beta_k 2^{-s_k}) + \sum_{k>q} \beta_k \leqq \sum_{k\leqq q} 2^{k/2-q+2} + \sum_{k>q} \beta_k \leqq \alpha_q. \qquad \square$$

PROPOSITION 1. *X contains no subspace isomorphic to* c_0.

In view of [3], Theorem 1.c.4 it suffices to show that every norm-bounded increasing sequence h_n has a limit in norm. In view of Lemma 3, it suffices to show that if $h_n \in \mathscr{C}$ is an increasing sequence, then $h = \sup h_n \in \mathscr{C}$. By definition of $\mathscr{C}$, for each n we can find $g_n \in \mathscr{C}'$ such that $\| (h_n - g_n)^+ \|_1 \leqq 2^{-n}$. It follows easily that $h \leqq \liminf_{n \to a} g_n$, so that $h \in \mathscr{C}$. □

To complete the proof of Theorem 2, it remains to show that E contains no copy of L^1. As already explained, in view of the results of [2], it suffices to show that

$$\lim_{t \to \infty} t \sup_{g \in \mathscr{C}} |\{g \geqq t\}| = 0$$

or even

$$\lim_{t \to \infty} t \sup_{g \in \mathscr{C}'} |\{g \geqq t\}| = 0. \tag{5}$$

For $g \in \mathscr{C}'$ we have $g = \Sigma \beta_k g_k$ where $\Sigma \beta_k \leqq 1$, $g_k \in \mathscr{C}_k$. By definition of $\mathscr{C}_k$, for each q we have $g \leqq T(g_1) + g_2$ where $\| g_1 \|_1 \leqq 2^q$ and $\| g_2 \|_1 \leqq 2^{-q}$. Since

$$\{g \geqq t\} \subset \{T(g_1) \geqq t/2\} + \{g_2 \geqq t/2\},$$

we have

$$t\,|\{g \geqq t\}| \leqq t \sup\{ |\{T(f) \geqq 2^{q+1} t\}| : \| f \|_1 \leqq 1\} + 2^{-q+1}$$

so that (5) follows from Theorem 1. The proof is complete.

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